

Learn log

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I present a collection of questions that I have encountered through my PhD. From obvious to not so obvious! I hope to be able to make a tool for researchers along this line, so that they can have their own glossaries — because, no answer is better than the answer you give to yourself! Hopefully, ultimately powered by AI.

1. ANSWERS

1. What is differential rotation of a star?

Differential rotation is seen when different parts of a rotating object move with different angular velocities (rates of rotation) at different latitudes and/or depths of the body and/or in time. This indicates that the object is not solid. In fluid objects, such as accretion disks, this leads to shearing.

In stars, the zonal wind and meridional circulation together constitute differential rotation and is usually caused by convection arising from steep temperature gradients.

2. What does helioseismology tell about the differential rotation in the Sun?

The radiative zone rotates almost like a rigid body while the convective zone has differential rotation.

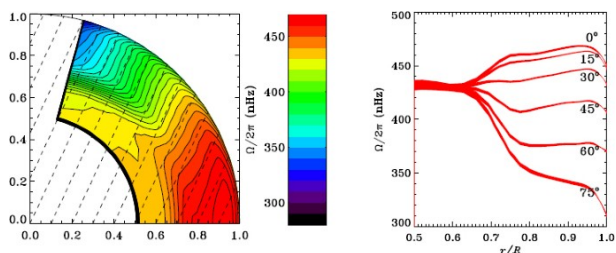


FIG. 1: Differential rotation profiles in the Sun.

3. Why is there no advection of zonal momentum in the meteo equations?

In meteo equations, there is no vertical fluid motion except for the ones that are mechanically or thermodynamically driven. So no vertical advection of momentum (it exists but can be neglected — hydrostatic approximation). This is discussed in the Showman review or Vallis textbook.

In the plane perpendicular to the vertical there is geostrophic balance (Low Rossby number flow) and that allows us to neglect inertia compared with rotation. Thus there are no momentum advection terms.

4. What is angular momentum transport in stars?

When a star evolves the core shrinks and the outer layers expand. For angular momentum to be conserved locally, the core must spin much faster than the outer layers which must actually slow down due

to expansion. This gives rise to shear between layers which transports angular momentum within the star.

5. What are the mechanisms for angular momentum transport in stars?

Reynold's stress: correlation between r and ϕ velocities. Caused by gravity waves propagating outwards carrying angular momentum or meridional circulation.

Viscous stress: caused by shear instabilities leading to turbulence which contributes to viscous stress by means of turbulent viscosity as molecular viscosity is almost negligible.

Maxwell stress: caused due to magnetic fields.

Purely hydrodynamic models (shear instabilities, meridional circulation) do not satisfy helioseismology. Magnetic effects can cause braking and transport of angular momentum.

6. What is Eddington-Sweet circulation and time?

The Eddington Sweet circulation is another name for meridional circulation in stars only driven by thermal anisotropies.

the regions it passes through.) Thus the circulation near the surface is from the poles to the equator.

5. *Summary.*—The unsymmetrical flow of radiation in a rotating star, by its **unequal heating effect**, maintains currents circulating in the meridian planes; but such currents are limited by the condition that the vertical velocity does not exceed

$$v_r = q\epsilon_1/ag,$$

where ϵ_1 is the mean rate of generation of energy (per gm. per sec.) interior to the point considered;

FIG. 2: Eddington's description.

However, the preceding statement “the circulation near the surface is from the poles to the equator”, is not quite correct. Helioseismology suggests otherwise.

7. History of meridional circulation (Talon S 2005).

Meridional circulation was first introduced in stellar astrophysics when von Zeipel (1924) studied rotating stars, and showed that they cannot be in thermal equilibrium. Eddington (1925) and Vogt (1925) concluded that this must induce a large flow, which would lead to some kind of mixing. This flow was calculated by Sweet (1950) a few years later. However, this classical formulation which is known as the Eddington-Sweet circulation has two major problems:

- The velocity field does not satisfy the conservation of angular momentum;
- It does not satisfy free-slip boundary conditions (in fact, the flow even diverges at the convective core boundary).

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Theoretical solutions have been proposed to those problems (e.g. Tassoul and Tassoul 1983, 1992; Zahn 1992), which in particular consider the advection of momentum by the circulation, as well as the concomitant effect of viscous diffusion. However, those solutions have a limited range of applicability.

8. How does Showman's meteo equation capture what happens in the Sun?

In the most idealized case the brunt frequency is enough to capture the steady-state effects of heating by means of a temperature stratification. The atmosphere of a planet is heated from outside while a star is heated from inside. But the atmosphere of the Earth is actually indirectly heated by radiative transfer from the heating of oceans and land by solar radiation. So the heat sources are kind of internal for both.

9. Why do we not care about the evolution of density of the stellar radiative material?

The density is determined from the equation of state $p = \rho RT$. The anelastic approximation is used and it removes the time derivative of density from the continuity equation, ruling out sound waves.

10. Is the propagation of sound waves adiabatic?

Yes! We do not need the temperature equation to solve for it. Sound waves are purely mechanical and disturbances travel fast, leaving no time for heat exchanges. However, very high frequency waves, could have sharp pressure/temperature gradients, resulting in heat transfer and loss of adiabaticity.

11. What is potential temperature?

Potential temperature of a parcel at a particular height in the atmosphere is the temperature that it will attain if brought to the ground adiabatically i.e. its pressure is increased to 1 bar without allowing any exchange of heat.

12. What kind of temperature profiles are stable and which are unstable? What are the equivalent potential temperature profiles?

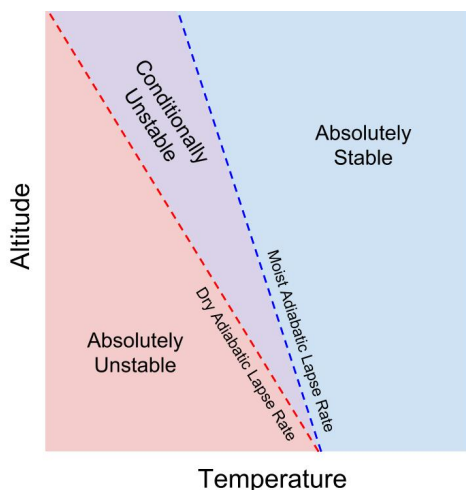


FIG. 3: Stability of an atmospheric profile.

For a dry atmosphere, a potential temperature profile constant with altitude would be neutrally stable and be at the threshold of convection. If potential temperature decreases with height (-ve slope) then the atmosphere is stable.

The temperature profile corresponding to a dry adiabat (constant pot. temp.) is neutrally stable. It has a -ve slope. Any temperature profile with a greater slope is unstable to convection. The moist adiabat has a more positive slope than the dry adiabat.

13. What is steep and what is flat for negative and positive slopes? What are the implications of steep and flat temperature profiles?

The more negative the lapse rate, the more stable is the atmosphere. Remember that the prime condition is hydrostatic balance, which says that pressure at the bottom of an air-column is equal to the weight of it. So pressure decreases with height. Colder air seems intuitively heavy, and hence lower temperature above higher temperature appears unstable, but note that the pressure also decreases with height, thereby reducing density significantly!

14. Why do we neglect the unsteady term in the energy equation for our 1D formulation?

$$\frac{\partial \tilde{T}}{\partial t} + \frac{N^2}{g} \frac{T}{\rho r^2} \tilde{\Psi} = \frac{1}{\rho C_p r^2} \frac{\partial}{\partial r} \left(\chi r^2 \frac{\partial \tilde{T}}{\partial r} \right). \quad (4.7)$$

After a transient phase, which lasts about a thermal relaxation time, $h^2 \rho C_p / \chi$, the evolution slows down and a balance is struck between advection and buoyancy in the heat equation. Then the time derivative in (4.7) may be neglected, as can be verified in the results of this section. With this simplification, the elimination of the perturbations of the thermodynamical variables yields an expression for the stream function in terms of the differential rotation:

$$\tilde{\Psi} = - \frac{2\Omega g}{N^2 C_p T} \frac{\partial}{\partial r} \left[\chi \frac{\partial}{\partial r} \left(\frac{r^2 P T}{\rho g} \frac{\partial}{\partial r} \frac{\tilde{\Omega}}{P} \right) \right]. \quad (4.8)$$

FIG. 4: Spiegel and Zhan 1992.

Because we assume thermal relaxation transients are fast compared to the other processes we are considering.

Because the temperature gradients following the mechanical/pressure/geostrophic/hydrostatic/adiabatic adjustment are initially strong (varies significantly over a small length scale), the thermal adjustment occurs very fast. Thermal adjustment would take the usual Kelvin-Helmholtz or the thermal timescale of τ_R if the temperature gradient due to mechanical adjustment was distributed all over the star; but our case is initially restricted to a shallow layer the depth h_0 of which depends only on rotation and stratification.

Now, following an initial, rapid adiabatic response on the dynamical time-scale ($\approx \Omega^{-1}$) across a depth $h_0 = 2\Omega r_0 / \lambda_2 N$ (§3),

FIG. 5: Spiegel and Zhan 1992.

15. Some thoughts on thermodynamics.

When only heat is supplied, the entropy of the system increases. For solids there is no change in volume and hence there is no pdV term, so heat addition translates to a change in temperature only. For a fluid that is not true and hence a lot of co-dependent parameters also change, like pressure or volume (density) or both, based on the ideal gas law.

I feel thermodynamics has two sides. One dealing with the energy contained in, added or subtracted from a system and the other that represents that in the form of physically measurable quantities like pressure, temperature and density. The first is the first law and the second is the ideal gas law.

16. What is the relationship between stratification and vertical motion? How do you describe the dynamics of an unstably stratified atmosphere — is it convective — if it is convective, then does the steady state convection maintain a time-independent stratification? Do the air parcels move adiabatically through it?

A stratified system is stable to vertical perturbation and is possible under the effect of gravity. Vertical motion is inhibited, unless there is a mechanical or thermal forcing.

Unstable stratification amounts to convection and the temperature profile remains adiabatic in the convective region. See outer sun, 0.7-0.9 in figure 14.

17. Why do we talk about motion in the radiative zone if there is stable stratification?

Fluid motion in stars can either be momentum driven or thermally driven. Although, a stable stratification in the radiative zone suppresses thermally driven motion, there is shearing from the outer convective layer (for near solar mass stars) which inputs momentum in the radiative zone. This can set up meridional circulation and the circulation in-turn affects the temperature distribution T '.

18. Why is the diffusion term in conservation laws replaced by a linear relaxation based on some time scale?

Diffusion is not replaced by a linear relaxation. Both of them are independent processes. In the atmosphere molecular diffusion is low and hence it is neglected. The linear relaxation is due to drag created by eddies which acts as a source or sink of energy, molecular weight, momentum or any other conserved quantity (variable). There is a time scale associated with this relaxation. This is the time taken by the variable to reach a steady state (u_F, μ_{ref}) exponentially.

19. What are relaxation times? What is the effect of τ_F and τ_R in our 1D equations?

Read the full answer please. It does not contradict the previous answer.

Relaxation time is the total time taken by the variable to be forced to a certain state. For example, the top layer is forced to 100 m/s in $10e7$ s, which is a relatively small time scale compared with the bottom layer where the same term acts to implement the equivalent of diffusion of momentum; From preliminary dimensional arguments we can find this timescale (from diffusion coefficients and a length-scale) and use it on a term like $-u/\tau_F$ to approximate diffusion (having more operators in the system consumes more computational time to solve it, a Laplacian is second order on top of that, so we use just a linear term).

20. What does the diffusion operator do?

It smooths out the gradients in the field, or if there are approximately positive and negative values of a certain field, then it takes all values to zero. The zonal velocity perturbation in the lower layer is one

such quantity and hence we force that velocity to 'zero'.

The frictional relaxation timescale or τ_F can be obtained from viscosity of the medium, because viscosity plays the role of smoothing out gradients i.e. bringing gradients to zero. However, in our case molecular diffusion is really small and only eddy diffusion acts to make things uniform. Just like viscosity is a drag like thing, so is eddy viscosity, or magnetic drag etc. They all apply brakes on momentum.

The time dependent part of the solution to the diffusion equation has the same form as the decay ODE i.e. both are exponential. The spatial part is tackled by u_F or T_E which are what the profile is finally forced to.

Similarly, temperature and mean molecular weight also diffuses i.e. is forced to a certain mean value (here 0).

21. Should the vertical tracer advection term in pressure coordinates be -ve or +ve when on the left side of the equation?

Positive on the left. Because, -ve and -ve makes +ve. The first one is the definition of $w(= dp/dt)$ which is -ve and the second -ve is due to $\partial/\partial p$.

When shifted to the right hand side of the equation, it is -ve.

22. What is the significance/meaning of the linear relaxation and diffusion terms in a partial differential equation?

The linear relaxation acts to bring the value of the dependent variable to a particular value at infinite time. The solution is exponential. It is just like radioactive decay, only difference is that the value of the dependent variable can also be negative, in which case it's value is raised to whatever is the particular value.

While linear relaxation acts on all length scales equally, the diffusion term acts strongly on sharp gradients (small length-scales), while it is not very effective at larger ones (weak gradients). The time dependent part of the solution to the diffusion equation is again an exponential decay, because it is variable separable. This comes from the one initial condition. The space dependent part is a sin or cosine or a linear superposition of them, the coefficients of which come from the two boundary conditions.

Also diffusion is irreversible and leads to an increase of entropy according to second law of thermodynamics. For a diffusive system, disturbances travel at infinite speed (like the heat equation, for which we need a relativistic heat equation), while for hyperbolic equations there is a finite speed of propagation of a disturbance and the results are time reversible.

23. I am not sure I understand the implementation of the momentum forcing: at the top boundary what is the condition for zonal velocity? Is it not constant? The code apparently does not have any condition for u at the top. I understand that the top layer (0-5 bars) relaxes to u_F two orders of magnitude faster than the bottom layer goes to zero. Both layers start off with zero initial u .

The domain has initial conditions of zero for all perturbations. The u velocity must relax to u_F in the top layer because it is *forced* and the timescale is given by τ_F and in the bottom layer there is basically a drag because the velocity must always go to

zero in a timescale determined by τ_F in that layer. When N is high, there is no penetration of circulation and hence the top layer is not dragged by the bottom layer and is able to attain the velocity of forcing u_F . The u velocity remains sharp. When N is low, the circulation penetrates deep and cause the bottom layer to pick up velocity from the top forcing u_F but the top layer cannot reach that value because the bottom layer now drags the top layer. So there is a more uniform u velocity profile across the top and bottom layers.

24. If μ' grows big why should $\nabla\mu$ be constant?

The total μ is a mean, plus a gradient, plus a perturbation. So long you neglect nothing, the perturbations can be as large as possible and it is not fundamentally/mathematically incorrect. But $\nabla\mu$ is just a background quantity.

25. Our stream function is one order of magnitude lower than Showman's. The three cells in temp develop early in the simulation. Circulation has just once cell. The value of horizontal velocity at depths is not zero.

The ipython notebook script that Kristen sent later has the correct order of magnitude of the streamfunction and also the correct slope at the bottom which matches Showman. Check the implementation of cumulative trapizoidal rule if required, in profile.py. There could still be some differences between Showman and our 2D version and the reason could be that Showman uses a sharp discontinuity, while we implement it through a smooth Heavyside function. Also, we don't force a cosine in the horizontal direction which Showman does. Three cells are because of $N < 10^{-6}$. The only boundary conditions are on w , v can take any value and so can u . The *condition* = "(ny! = 0)" is related to spectral modes and details of a spectral solver which I do not understand yet.

26. From Rauscher and Menou 2010: by nature of being a spectral code, instead of grid-based, we do not have any special treatment at the poles
27. What does — condition="(ny != 0)" — in "problem.add_bc("right(w) = 0", condition="(ny != 0)")" mean in the context of Dedalus2?

Nico: As in, I sort of know the reason, but it's not clear enough in my mind to provide an explanation here. It's about making sure the matrix to invert isn't singular. You must have a left(w)=0 somewhere, which would make the the BC on the right redundant for the 0th mode. So you write right(w)=0 only for all non-zero modes.

Check gauge conditions on Dedalus page, is it the same?

28. Taylor Proudman theorem? Taylor columns? Balance between rotation and stratification?

The TP theorem states that, for a geostrophic wind with NO variations in density/buoyancy/temperature in the horizontal plane, the vertical structure of the flow is uniform and there is no vertical velocity throughout the flow column as a result of no penetration at the bottom. Remember the expt. with the hockey puck in the slightly spun down rotating tank: the tank is slightly spun down to produce a very small flow, thereby lowering the Rossby number and ensuring geostrophy. The geostrophic flow avoids the puck and goes around it. Note that this would not be true

if there were density/buoyancy/temperature variations in the tank.

Also, because of the rotation, the flow gains some stiffness. This is best explained by keeping a buoyant ball of a similar size over the puck in rotating and non-rotating flows. The ball quickly rises to the top for the latter because fluid parting above the ball can move anywhere, which is not true for a rotating flow, because a Taylor column is formed and there is no exchange of fluid between that and rest of the flow.

If there is a variation in density/buoyancy/temperature in the horizontal plane in addition to geostrophy, it gives us the thermal wind. The horizontal variations break the vertical invariance of the column, resulting in a vertical gradient of the geostrophic velocity. NOTE: the vertical gradient is not due to friction at the surface of the Earth! In fact, there is the atmospheric boundary layer (mixed layer) above which geostrophic winds reside. Rather it is the equator to pole temperature gradient, which creates a vertical velocity gradient in the latitudinally oriented Jet Streams!

TP is broken if the Coriolis parameter is not constant i.e. in the beta plane.

29. Can Taylor columns form in stratified fluids?

A homogeneous fluid is bounded above and below by horizontal plane surfaces in rapid rotation about a vertical axis. An obstacle is attached to one of the surfaces, and at large distances from the obstacle the relative velocity is steady and horizontal. Solutions are obtained as power series expansions in the Rossby number, uniformly valid as the Taylor number approaches infinity.

If the height of the obstacle is greater than the Rossby number times the depth, a stagnant region (Taylor column) forms over the obstacle. Outside this region there is a net circulation in a direction opposite the rotation. The shape of the stagnant region and the circulation are uniquely determined as part of the solution.

Possible geophysical applications are discussed, and it is shown that **stratification renders Taylor columns unlikely on earth**, but that the Great Red Spot of Jupiter may be an example of this phenomenon, as Hide has suggested.

FIG. 6: Inertial Taylor Columns and Jupiter's Great Red Spot - Ingersoll

"Phillips (1963) gives equations appropriate to a stratified, inviscid ideal gas on a rotating sphere. These equations are based on an expansion in the Rossby number similar to that presented in this paper. The motion is still horizontal and nondivergent, to lowest order in the Rossby number, but now the lowest order stream function depends on the vertical coordinate. *This simply reflects the fact that the Taylor-Proudman theorem (10) does not hold for a stratified fluid, and so Taylor columns will not form when stratification is important.*"

<https://authors.library.caltech.edu/3988/1/INGjas69a.pdf>

Showman 2006: "Most importantly, the Taylor-Proudman theorem does not apply to statically stable fluids, which can support vertical shears of the zonal wind through thermal-wind balance (Holton, 2004, p. 70)."

30. What is the orientation of Taylor columns in the planetary scale?

Parallel to the rotation axis. See section on Jupiter in Vallis.

31. What is the difference between shallowness parameter and aspect ratio?

Shallowness parameter: vertical thickness of surface flow / radius of parent body

Aspect ratio: vertical thickness of flow / horizontal length of flow

32. What is the Prandtl number?
Ratio of molecular momentum diffusion (viscosity) by thermal diffusion (conductivity).
33. A great resource for basic stellar physics:
https://www.astro.ru.nl/~onnop/education/stev_utrecht_notes/
34. What do Spiegel and Zahn say about mean molecular weight gradients?
The authors show that the molecular weight gradient contributes similarly to the hyperdiffusion term and hence does not prevent penetration of the meridional circulation. A hyperdiffusion term is similar to the diffusion term in effect but usually gets reversed in sign as the order of the term moves from 2 to 4 to 6 to 8. The higher-order terms are more effective when the initial profile is sharp.
See the other section on hyperdiffusion.
35. What are the random perturbations in the simulation for?
They are for solving turbulent flows. But you also need nonlinear advection terms and higher resolution to simulate turbulent flows.
36. What is scale height H ?
 H is known as the scale height of the atmosphere and can be locally defined for regions where the temperature is fairly constant as RT/Mg . R is a constant. The higher the T , the more fluffy the atmosphere is and hence H is greater. It becomes less fluffy for a heavier gas i.e. high M and greater gravity which pulls and squishes everything together reducing H . The scale height basically defines how rapidly atmospheric pressure decreases with altitude.
Scale height is commonly used to describe the atmosphere of a planet. It is the vertical distance over which the density and pressure fall by a factor of $1/e$. These values fall by an additional factor of $1/e$ for each additional scale height H .
37. What is the significance of unsteady terms in our equations and what can be said about their magnitudes?
It is all about focusing on long-timescales ($\partial u/\partial t$), which makes us neglect the thermal relaxation ($\partial T'/\partial t$) and the thermal wind adjustment (relaxation to geostrophic balance—same as geostrophic adjustment, $\partial v/\partial t$). But, for the 2D setup it we do not neglect the thermal relaxation.
38. What are the different sources of gravity waves in the atmosphere?
Orography, convection, geostrophic adjustment. These are small scale phenomena and hence are not captured by GCMs. The momentum equations can be supplemented with a drag term that accounts for gravity waves depositing energy to mean flow via breaking (phase speed of the wave matches the mean flow, breaking happens).
39. Why is there no meridional component of the momentum forcing in our system even though the convective zone has full 3D circulation?
Meridional circulation in the convective zone is way weaker than the zonal circulation.
40. What do different values of μ_{min} mean in our simulations?
Different composition of the atmosphere, which can be true for exoplanets. Remember Barium was found recently in one of the atmospheres.

41. Why does Jupiter spin faster than other terrestrial planets?
Planets are formed from a protoplanetary disc which is an outer swirling cloud of gas around a star. Now due to initial density anisotropies certain regions accrete more material than others. If there is some initial spin to this local blob of gas, it is likely that it will spin faster and faster as it accretes more and more mass. And as it does so, it's gravitational attraction increases and it pulls in more stuff, increasing it's angular momentum even further. Most heavy objects in the universe are spinning faster than relatively lighter objects. Terrestrial planets are way smaller than gas giants and hence have smaller spins.
Jupiter has non-uniform ammonia distribution. Our theory may be useful but not much because, Jupiter has other chemical reactions and phase changes going on. Fact: Jupiter would be a brown dwarf, if it was slightly heavier. A brown dwarf burns deuterium through nuclear fusion.
42. Discussion with Eesha on stars.
- (a) Hiyashi track - so long a star is not born, it remains as a proto-star which is contracting because of gravity, getting compressed, leading to a rise in temperature, but that is nullified by loss of energy through radiation; no hydrogen burning; timescale for this is the Kelvin Helmholtz timescale; vertically downwards on the HR diagram collapsing on to the main sequence.
The energy source is gravitational potential energy. While the total luminosity is comparable to the solar value, the proto-star stages have the KH time scale. They are also fully convective and thus chemically homogeneous.
- (b) Nuclear timescale is so long as it burns H which is much greater than τ_{KH} and the dynamical timescale (which is basically how fast a star collapses when there is no pressure inside; this one is the fastest); rotational timescales are longer than τ_{nuc} and hence rotational effects can be considered perturbations to an axisymmetric base state
- If the spreading is laminar, we may ignore the viscous term in (4.9) or (4.10). We can look for a similarity solution of the resulting equation with the region of influence of the convection zone spreading like the $1/4$ power of time:
- $$h \approx r_0 \left(\frac{t}{t_{ES}} \right)^{1/4}, \quad (4.12)$$
- where t_{ES} is a local Eddington-Sweet time given by
- $$t_{ES} = \left(\frac{N}{2\Omega} \right)^2 \frac{r_0^2}{\kappa}.$$
- Such a regime is familiar in the stratified spin-down problem (e.g. Sakurai 1970; Spiegel 1972). In the Sun, $t_{ES} = 2.2 \cdot 10^{11}$ years and $r_0 = 480,000$ km; thus $h \approx 200,000$ km after $4.6 \cdot 10^9$ yrs.
- FIG. 7: S&Z Eddington-Sweet time.
- (c) efficient angular momentum transport would imply slower core rotation rates, but MESA models cannot do that
- (d) protostellar disc accumulates from molecular gases around the protostar and the outer part

of it is the circumstellar disc or protoplanetary disc where planets form

- (e) Fe is the heaviest element a star can fuse, because it has maximum binding energy (to break He into H, we need to provide energy and hence it gives us energy when H fuses to form Fe, binding energy reduces when atomic number increases after Fe), look at the onion structure of high mass stars
Heavier elements may be manufactured in AGNs which are death beds of generation one stars (one that only had H to begin with)
- (f) red-giants burn H in their shells and not in their core
- (g) the initial mass of stars determine what they will become when they die in a supernova, white dwarfs (electron degeneracy pressure, white dwarfs don't collapse because they are smaller than the Chandrasekhar limit), neutron stars (neutron degeneracy) or black holes, these are all compact objects
- (h) Heavy stars obey CNO cycle for H fusion because they have these heavy elements, while small stars go via the P-P chain
- (i) TESS and Kepler told us about differential rotation of stars, they dove deep into stars, before that there was no information and people did mostly empirical and theoretical stuff without any observational backing

43. Rossby number for the sun?

Can be expressed as $\Delta\Omega/2\Omega$. At the surface, the sun rotates faster at the equator (25 days) than at the poles (35 days) which implies differential rotation. $\Delta\Omega$ is the deviation from background solid body rotation.

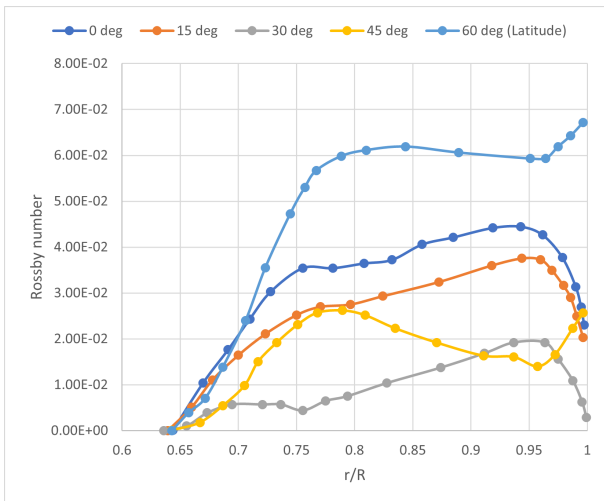


FIG. 8: Rossby number in the Sun.

44. What is Rossby deformation radius?

Rossby radius of deformation NH/f is the length scale at which rotational effects become as important as buoyancy or gravity wave effects in the evolution of the flow about some disturbance.

45. Solid body rotation versus Kepler rotation.

Solid body rotation is the same as a steady-state forced vortex while Kepler rotation is like a free vortex. For the latter, the closer one is to the centre,

the faster is the rotation. The opposite is true for the former.

For Keplerian rotation, there are important effects like, shear, geostrophy, hydrostatic balance.

46. Boussinesq approximation. What is it?

Density variations are important only in the buoyancy/ gravity term. Temperature differences change the density but not so much as to affect inertia, viscosity or pressure terms. Mostly used for the oceans.

47. Scale height $H = RT_0/g$. For a stratified atmosphere, $dT/d\ln P$ is non-zero. Does it mean scale height is also a variable or T_0 is a mean temperature and H is therefore a constant?

In detail, H is a constant only in a isothermal atmosphere. In general T_0 varies with pressure, but it could mild enough (close to isothermal) that H is nearly constant. We can choose what atmosphere background profile we model in our case..(isothermal or not).

48. Why does it snow even when surface temperature is greater than 0 degrees?

All precipitation starts off as snow but melt as it comes down. In order to melt to form droplets it needs sufficient warmth and time. Else it remains snow even if temperatures are greater than zero. Also, a minimum amount of moisture is important for holding snow intact, it melts easily in dry areas.

49. What is the convective mixed layer?

Ocean: The surface mixed layer is a layer where turbulence is generated by winds, surface heat fluxes, or processes such as evaporation or sea ice formation which result in an increase in salinity.

Atmosphere: The mixed layer, the major part of convective boundary layer, has a nearly constant distribution of quantities such as potential temperature, wind speed, moisture and pollutant concentration because of strong buoyancy generated convective turbulent mixing.

Beyond this zone there is free atmosphere where winds are mostly parallel to isobars i.e. geostrophic. Clouds appear at the top of the convective boundary layer, which is also called the lifting condensation level.

The vigorous turbulence can be caused by either strong winds or wind shears that generate mechanical turbulence (called forced convection), or by buoyant turbulence (called free convection) associated with large thermals. The buoyantly generated mixed layers are usually statically unstable, caused by heating at the bottom boundary such as the earth's surface or radiative cooling at the tops of cloud or fog layers within the mixed layer. The terms mixed layer, convective mixed layer, and convective boundary layer commonly imply only the buoyantly stirred layer.

50. It is diffusion again: molecular diffusion versus eddy diffusion; momentum diffusion versus mass diffusion!

I was implementing anisotropic diffusivities in our 2D Dedalus code and had a rendezvous with different kinds of diffusion in the context of flows and heat transfer.

Firstly, Kirsten: the diffusion operator can be expanded in Taylor series to obtain the linear relaxation operator based on a diffusion timescale, but that is just first order and hence is not very accurate

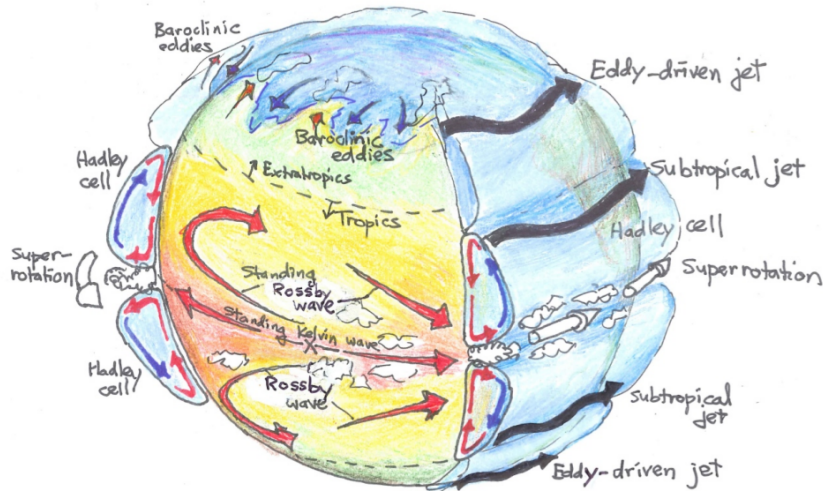


Fig. 2.— Schematic illustration of dynamical processes occurring on a generic terrestrial exoplanet. These include baroclinic eddies, Rossby waves, and eddy-driven jet streams in the extratropics, and Hadley circulations, large-scale Kelvin and Rossby waves, and (in some cases) equatorial superrotation in the tropics. The “X” at the equator marks the substellar point, which will be fixed in longitude on synchronously rotating planets. Cloud formation, while complex, will likely be preferred in regions of mean ascent, including the rising branch of the Hadley circulation, within baroclinic eddies, and—on synchronously rotating planets—in regions of ascent on the dayside.

FIG. 9: Circulation in the atmosphere of terrestrial exoplanets [1].

when the flow has smaller scales involved in it. The diffusion operator acts strongly when there are sharp gradients in Heat, Momentum, or passive Tracer concentration (abb. HMT) and less so when the gradients are flat. Hyperdiffusion, which is basically fourth or sixth-order derivatives of HMT diffuses more on small scales and less on large scales than regular diffusion, and therefore it prevents crashes due to sharp gradients in spectral codes. It is used as a means of removing small-scale features and the higher the power, the stronger the scale separation. The linear relaxation acts only on one scale, given by the dominant length scale or the pressure scale height H . The diffusion acts on different scales; hyperdiffusion on small scales is stronger than on large scales.

Chris: Hyperdiffusion also causes inverse diffusion under certain circumstances. This is related to turbulence modeling and the inverse cascade.

The diffusion operator kind of brings equality everywhere and smooths out gradients.

Next up, molecular diffusion is because of the random motion of the molecules themselves and not because of the flow velocity. When treating turbulent flows ie. high Reynolds numbers, as laminar flows, one needs to account for the mixing/energy-dissipation brought about by fluctuations in flow about the mean. On the whole the flow might not be going anywhere, but locally there can be random motion called eddies which transfer energy by breaking down in size and finally reaching Kolmogorov length scales where molecular diffusion takes over. Turbulent flow has much greater viscosity and this is expressed as *turbulent/eddy* viscosity. Naturally, the coefficient of this kind of viscosity will be dependent on the velocity fluctuations in the flow. It also depends on a length scale called the Lagrangian length scale: over what distance does the velocity

fluctuation have a significantly high autocorrelation ie. correlated with itself within a certain time called the Lagrangian time scale.

$$\epsilon = \tilde{u}L_L \quad (1)$$

https://www.astro.princeton.edu/~burrows/classes/542/papers/Alwin.Roberts_Turb_Diffusion-1.pdfhere

In the context of our problem, horizontal length scales / velocities are much stronger compared with vertical ones (8 orders of magnitude). Therefore the eddy diffusivities must also be anisotropic.

Further, diffusivities of heat, momentum and passive tracers are different. The Prandtl number and Schmidt number are ratios of these diffusivities. This is true irrespective of turbulence or eddy diffusivity.

Pr = momentum d. / thermal d., and

Sc = momentum d. / tracer mass (molecular) d.

51. Convolution: Inverse Fourier transform of the product of $f(t)$ and $g(t)$ is the convolution of $f(t)$ and $g(t)$ ie. $f(t) * g(t) = \int f(\tau)g(t - \tau)d\tau$.
52. Seismic waves traveling through anelastic media gets not only attenuated but also physically dispersed.
53. Acoustic waves. Seismic waves. Smaller frequencies die out first. So we hear only high frequencies across a wall. Also, sound waves are non-dispersive. So a person sitting at the back of the auditorium listens to the exact same frequencies as someone at the front, albeit with higher frequencies damped slightly more. This is because all frequencies travel at the same speed ie. speed is independent of frequency. Group speed and phase speed are the same for a non-dispersive medium.
54. What are group and phase speeds?

Phase speed is ω/k . This is the speed of crests and troughs.

Group speed is $\partial\omega/\partial k$. The speed of propagation of the envelope or the energy associated with a superposition of waves.

55. What is the instantaneous frequency of a random signal whose amplitude and phase are known functions of time?

$f_{inst} = \frac{1}{2\pi} \frac{\partial\phi}{\partial t}$, where, ϕ is the phase.

56. Why is the shallow flow theory valid in the radiative zone of a star? Is our theory a shallow flow theory, the primitive equations?

Shalowness is not valid. But hydrostatic balance is applicable still.

57. Mechanical pressure versus thermodynamic pressure!

The mechanical pressure is a measure of the translational energy of the molecules. The thermodynamic pressure, on the other hand, is a measure of the total energy, which might include additionally vibrational and rotational modes and, for liquids and dense gases, intermolecular attraction.

58. Gyroscopic pumping (Ekman pumping) is relevant for our problem. There is a paper on it too: <https://ui.adsabs.harvard.edu/abs/2010ApJ...719..313G/abstract>. What is gyroscopic pumping?

Convergence and divergence (or circulation generated in) of fluid due to retrograde and prograde rotational forces, respectively. Ekman pumping is a special case when friction (wind stress at the surface of the ocean) is responsible. See this video: <https://www.youtube.com/watch?v=p0KwnYsEaok>. Ekman flow is also the reason ice-chunks moves perpendicular to the direction of the wind. Coriolis force is the prime driver of this kind of phenomena.

59. Nuclear fusion inside stars: not classical, but quantum!

Classically particles just have one velocity. Quantum description treats it as distribution peaking at the classical velocity/momentum. Even though the classical momentum is not enough to overcome the Coulomb barrier inside stars, the small probability of particles having velocities larger than the classical one is sufficient to assume that they could *tunnel* through the coulomb barrier, giving rise to fusion like conditions.

60. What are the different timescales for the sun?

10.7 A Hierarchy of Timescales

So if we arrange our timescales, we find a strong separation of scales:

$$\begin{array}{ccccccc} \tau_{nuc} & \gg & \tau_{KH} & \gg & \tau_{\gamma,diff} & \gg & \tau_{dyn} & \gg & \tau_{\gamma} \\ 10^{11} \text{ yr} & \gg & 3 \times 10^7 \text{ yr} & \gg & 10^4 \text{ yr} & \gg & 30 \text{ min} & \gg & 10^{-10} \text{ s} \end{array}$$

FIG. 10: Nuclear, thermal, photon diffusion/ radiative, dynamical, matter-radiation equilibrium.

Nuclear - time to exhaust all fuel at constant burn rate
 Thermal - time to radiate away all the gravitational binding energy at current luminosity
 Photon diffusion/ radiative ($N * l/c$) - time taken by photons to reach from core to surface via random walk of N steps where $N = R/l$; l is the mean free path of photons through the gas
 Dynamical - time taken to collapse if pressure support is removed / time

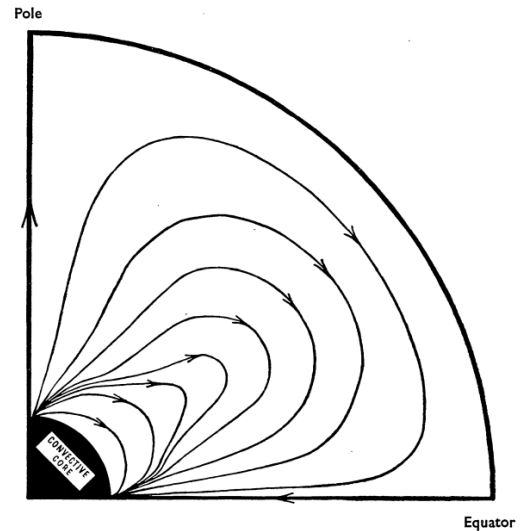


FIG. 1.—Circulation in a uniformly rotating star.

The total time taken for material to travel from the core of the Sun to the surface along the axis of rotation is given by

$$t_{\odot} = \int_{r_{\text{core}}}^R \frac{dr}{[v_r]_{\theta=0}} \sim 8 \times 10^{12} \text{ years.}$$

FIG. 11: From Sweet 1950.

taken to explode if gravitational support is removed
 Matter-radiation equilibrium (l/c) - mean free path of photons by speed of light

http://www.mit.edu/~iancross/8901_2019A/lec011.pdf

There is also Eddington Sweet time (τ_{ES}) = τ_{KH}/π where $\pi = \Omega^2 R^3/GM$. This is the time required for the redistribution of angular momentum due to meridional circulation. Usually of the order of 10^{12} years for the sun. But relatively smaller for massive stars.

61. Molecular weight gradient has the same timescale as nuclear timescale. Eddington Sweet timescale is 10^{12} yrs while mol. wt. grd. tmcs. is 10^{11} yrs.

62. Why is the hydrostatic approximation valid for our system? S and Z 1992 starts with the fact that hydrostacy is broken in stars...

Hydrostatic balance is important for both geophysical and stellar flows. For the latter, it implies that, there can be no radial velocity when treated in spherical coordinates; however, for GFD, it is okay to have hyd. bal. and still have small vertical motion. S&Z possibly talk about the spherically symmetric hydrostatic balance that is relevant for stars.

63. We had some contention on the timescale for geostrophic adjustment. What was it?

Geostrophic adjustment is also called an adiabatic adjustment and happens on a dynamical timescale, which is a few times the rotation time of a Sun-like star.

64. Is the sun in geostrophic balance? Where does the assumption break down?

Check question 43. Usually, geostrophy breaks down close to the equator where f or the Coriolis parameter is small.

65. Why is there no pressure gradient in the zonal direction?

Because our model is axisymmetric.

66. In that case, what drives the zonal flow?

The convection zone drives the zonal flow. This is the case of “shallow forcing” in Showman, which attributes this zonal flow to moist convection or differential solar heating. For the Sun, solar winds drive the convection zone and the convection zone torques the radiative zone.

67. Why do our equations have the unsteady term but no advection terms, even though both of them represent inertia of the flow?

We cannot neglect time derivative of u because that will remove all dynamics. There is no separate evolution equation for v as the v -momentum equation gives geostrophic balance. The vertical momentum gives hydrostatic balance and hence there is no time evolution for w .

On the other hand, the advection terms are nonlinear and therefore smaller than the unsteady term because they have a product of two velocity perturbations in the numerator. These velocities are small compared to the rotation speed at the surface. The follow up question is what happens to the base state u, v, w velocity terms? Well, base state $v, w = 0$. The base state u is a constant because of solid body rotation.

68. Turbulent versus laminar simulations.

Also, nonlinear terms are important only for turbulent simulations at high resolution, for example, in convection zones. The radiative zone is stably stratified and hence laminar simulations are not only okay, but also cheaper. Radiative zones, however, can be subject to instabilities GSF, MRI, shear, Rayleigh-Taylor, gravity waves etc, which can effectively cause some turbulence, but their effect can be captured by calculating and appropriate diffusion coefficient, that can be added to the usual equations.

69. Convection and mixing are different processes. Woah wait, what?

Convection is *adiabatic* and *reversible*. It only implies fluid stuff moving around. It has a timescale depending on the driving force, thermal or mechanical. Convection helps move parcels of high temperature or tracer concentration into regions low in these quantities and vice versa, thus creating large local gradients. Diffusion can now act on these gradients and end up *mixing* everything together.

Mixing is *diabatic*. This also implies *irreversible*. It acts through molecular diffusion only. However, convection is adiabatic and hence reversible. Diffusion acts fast on strong gradients, which are not as much without convection. Diffusion follows the same physics as conduction ... so effectively, energy transfer can happen only by diffusion (molecules) and radiation (photons).

And then, to confuse you further, there is something called radiative diffusion lol! (Update: not so confusing any more; radiative diffusion is just the diffusion of heat via photons, rather than molecules which ‘conduct’ heat; both use the same mathematical operator, the laplacian of the temperature field; also radiative diffusion is much stronger than conduction and that is what happens inside stars)

70. How are meteo equations different from Boussinesq or Anelastic equations?

Meteo equations are a slightly simplified form of anelastic equations.

71. What holds more moisture? Warm or cold air? High pressure or low pressure air?

Warm air is further from the dew point than cold air, so holds more moisture. The relative humidity increases as the temperature is lowered and becomes 100 percent at the dew point.

High pressure holds more moisture on the other hand.

72. Moist air vs dry air; normal water vs saline water.

Moist air is lighter than dry air and hence rises to form clouds. Also, moist air has H_2O which displaces the heavier N_2 to make a parcel lighter.

Saline water is heavier and that is why we float easily in the ocean compared to a pond/lake. Salts are heavier and hence increases the density of water.

73. Why does the thermal wind have a detailed μ and the temperature equation not?

It is all for numerical convenience! We do not want to unnecessarily complicate the equations. Indeed, the mean molecular weight should be expressed with all its parts, $\mu = \mu_{min} + \alpha * \ln p + \mu'$, even in the equation for temperature. But, the variation in μ is, at most, by a factor of two through the sun. This variation can be numerically adjusted by modifying the Brunt frequency as in the ‘non-dimensionalising’ notes. We know what the effect of the change of brunt frequency on the flow is! Why complicate that term then?

In the μ conservation and thermal wind equations we have gradients of μ' rather than its absolute value. The absolute value is low but the gradient can still be large and have significant effect on the flow. So we keep those terms. Should be evident from dimensional analysis.

74. Why don’t we have T_E in our temperature equation?

Because, T_E is for latitudinal variation. The vertical variation of T is included in the Brunt frequency term. T_E would be important if we were to apply thermal forcing, like that on a planet or the Earth i.e. establish baroclinicity by imposing an equator-to-pole temperature gradient. The usual Eddington-Sweet circulation can also be generated by applying T_E ; in fact, that is the physical mechanism behind it. Rotation creates isentrops/isopycnals/isotherms which are not aligned with isobars (baroclinicity) giving rise to a meridional circulation.

75. What are the assumptions and limitations of our problem setup?

- The flow is zonally symmetric i.e. quantities do not vary with the azimuthal angle (or longitude), ϕ .
- f-plane (see figure 12a): We use Cartesian coordinates rather than spherical. The f-plane approximation uses a fixed value of the Coriolis parameter $f (= 2\Omega \sin \theta)$ set to correspond to mid-latitudes ($\theta = \theta_0$). Here Ω is the background rotation rate.
- The horizontal extent L_y , is close to the entire length from the pole to the equator comparable to the thickness of the star’s radiative zone. This stretches the f-plane approximation beyond the formal limit but captures the proper horizontal scale of interest.
- The pressure scale height H is assumed to be constant, as would happen in an isothermal

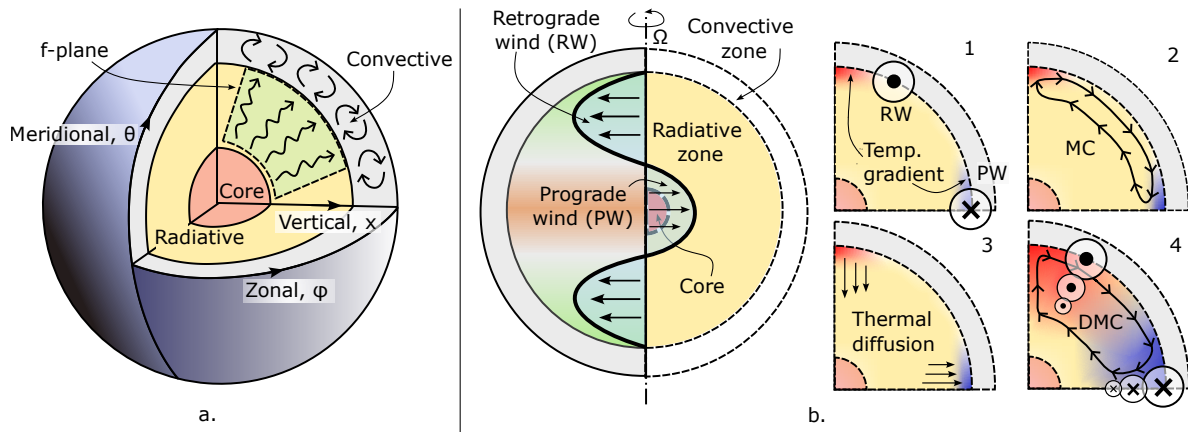


FIG. 12: a. The interior structure of a solar-type star showing the convection zone, the radiative zone, the core and the principle coordinates. b. Schematic of deep meridional circulation (DMC) inside the radiative zone of sun-like stars. The panel on the left shows the external structure of the zonal forcing/differential rotation applied by prograde and retrograde winds. It also shows the typical internal structure of solar-type stars, including the core, the convection zone, and the radiative zone. The numbered panels, in order, show the sequential development of the DMC. The prograde and retrograde *zonal winds* are represented by arrows pointing out of and into the plane. The red and blue regions represent the baroclinic structure that results from the zonal torque following the thermal wind balance. The initial meridional circulation (MC) cell is shown as a loop with corresponding arrows. The final circulation cell is deep and corresponds to a smoother gradient of zonal velocity.

radiative zone. However, the latter is also stably stratified with a Brunt frequency N i.e. temperature changes with pressure, albeit with significantly small variations.

- Constant gravitational acceleration g ; for the sun this changes by a factor of 2 with depth in the radiative zone.
- Hydrostatic balance: this is a common assumption in both stellar astrophysics and geophysical fluid dynamics (GFD). It implies that vertical motion w is significantly suppressed compared with horizontal velocities u and v , which is a direct consequence of the aspect ratio of the flow λ being $\ll 1$. A deep radiative zone violates the thin layer approximation built in our formalism.
- Geostrophic balance: The Rossby number $Ro \sim \Delta\Omega/\Omega$ of the flow is small. Here, $\Delta\Omega$ is the differential rotation. Inertial forces are therefore dominated by rotation Ω .
- A consequence of the above two assumptions is that the timescale for adiabatic/geostrophic adjustment is of the order of $1/\Omega$ (dynamical) which is much smaller than the Eddington-Sweet circulation. Assuming that the slow meridional flow is in geostrophic balance is thus well justified.
- The hydrostatic balance and geostrophic balance combine into the thermal wind balance. This implies that a vertical gradient in velocity corresponds to a horizontal gradient in temperature.
- Angular momentum is conserved by meteorological equations.
- Diffusion of momentum and temperature are approximated by linear relaxations.
- The feedback of the radiative zone on surrounding convection zones is ignored.

76. What kind of PDEs are we solving? Is our fluid Newtonian? What are the physical quantities that diffuse and how are they implemented?

We are solving a non-linear initial value problem (IVP) in 2D. One may also solve linear IVPs using Dedalus. Linear problems can also be cast as eigenvalue problems, which allows us to check for instability and growths. There is a tutorial which first solves an EVP to identify the fastest growing unstable mode of a setup and then uses it as the initial condition to solve a non/linear IVP.

Our equations are most likely parabolic, just like the Navier-Stokes equations. Our fluid is also Newtonian, but viscous (not Euler's equations). Heat, momentum and mean-molecular weight (tracer) are all diffused via linear relaxations.

77. Numerics is an art:

10e16 — relative computations crash i.e 1e200 + 1e184 is not accurate but 1e200 + 1e190 is fine!

78. What is the temperature gradient in a convective layer? Is it constant throughout?

It should be roughly adiabatic.

79. How does temperature, density and pressure change with radius in the sun?

From MESA for 1 solar mass at 4.6 Gyr (Sun's age): Temperature is almost constant (at least in the radiative zone), hence iso-thermal! Pressure and density increase exponentially (linear with $\log p/\rho$) with depth, owing to hydrostatic balance and the ideal gas equation of state. Actually temperature also changes exponentially but the exponent is small. **This means the temperature scale height is much larger than the density or pressure scale height and thus temperature can be considered a constant!**

80. What is the y axis of our 1D paper plots?

It is in units of pressure scale height, which is linear because we have taken log of pressure which is exponential.

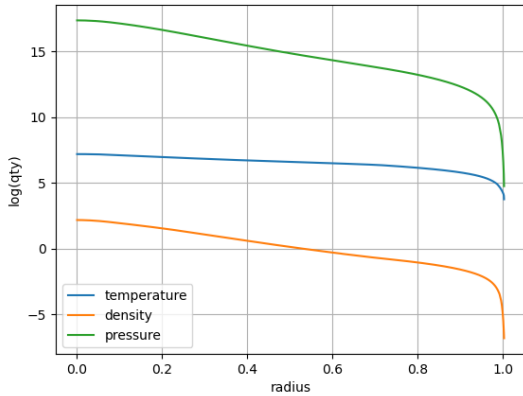


FIG. 13: Temperature, density and pressure profiles in the Sun.

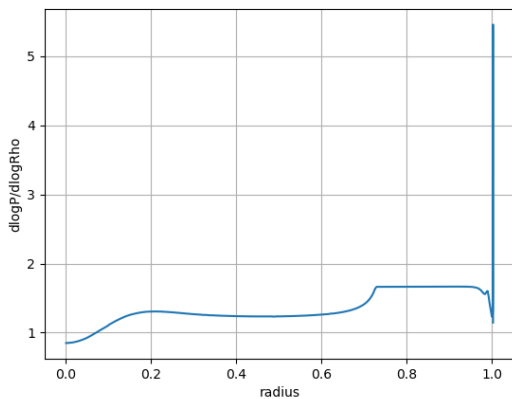


FIG. 14: Radiative and convective zones based on density gradient.

81. Where is the convective zone in the Sun?

The portion of the star where $d \log P / d \log \rho$ is constant is the convective zone. $d \log \rho / d \log P$, the inverse of $d \log P / d \log \rho$ represents the temperature gradient in the star. It is constant in the convective zone because convection doesn't allow the density/temperature gradient to increase beyond a certain value which is the adiabatic gradient, by continuously redistributing heat. Convection in itself is adiabatic i.e. when material moves around due to buoyancy, it does not exchange heat with the surroundings until it has reached a location after which it doesn't move any further; now the blob/material has enough time to exchange heat with the surroundings. It deposits heat, cools, becomes heavy and sinks back again and doesn't heat back up until it has reached mechanical equilibrium.

82. What is the difference between thermal and radiative diffusion? Do we have momentum diffusion in our problem? How is it captured?

See questions 69.

We have viscosity to diffuse momentum. It is captured using a linear relaxation $-u_F / \tau_F$, where τ_F is the drag timescale derived from a scale height and the coefficient of viscosity.

83. Eddington-Sweet circulation is baroclinic, means

that it is driven by temperature gradients. Our circulation is driven mechanically. Is ES circulation only relevant for the convective zone?

In the convection zone there is meridional circulation as well. Stuff flows towards the poles at the surface and towards the equator at the base of the convection zone. But, ES circulation is valid even in the radiative zone. The primary difference between the convective and radiative zone, so far as numerics are concerned is that the convective zone has significantly higher diffusivity, in order to capture the effects of turbulence.

84. Taylor-Proudman versus thermal wind!

The latter is due to a latitudinal entropy/density/temperature/buoyancy gradient (baroclinicity) and prevents the formation of Taylor columns. For TP, we need constant density (barotropic) and that makes rotational contours align with the rotation pole.

Can also be understood with respect to the ratio of Brunt frequency and rotation (Rossby radius of deformation), wherein, if N is large, it prevents the formation of Taylor columns (thermal wind structure/baroclinicity). In other words, large N prevents penetration of zonal forcing. While if rotation is stronger, TP is still possible.

85. Some ideas of magnetism inside a star.

The magnetic structure of the Sun is extremely complex and not like that of the Earth. Convective zone dynamo studies are linked to both differential rotation and meridional circulation (Choudhuri), while the Spruit dynamo for the radiative zone is mostly due to differential rotation. And then there are surface magnetic fields and solar winds which torque the convective zone establishing differential rotation via conservation of angular momentum because the fluid is attached to magnetic lines of force. Magnetic forces also apply drag called magnetic braking, which can be present everywhere.

86. The single most important paragraph from Garaud and Brummell 2009.

Next, we studied how much mixing these flows might induce in the underlying radiative zone. *In this quasi-steady formalism, we found that the magnitude of convection-zone-driven flows decays exponentially with depth below the radiative-convective interface on the lengthscale l_2 , where $l_2 = R_\odot / \text{Pr}^{1/2} b_{rz}$, as determined in Section 2.4.2 (see also Gilman & Miesch 2004 and Garaud & Brummell 2008). This penetration corresponds (in the linear regime) to a so-called "thermo-viscous" mode. The limit $l_2 \ll R_\odot$ corresponds to a strongly stratified limit, where the flow velocities are rapidly quenched beneath the convection zone. The limit $l_2 \gg R_\odot$ corresponds to the weakly stratified case, where the thermo-viscous mode spans the whole radiative interior and the stratification has little effect on the flow. It is important to note that "weakly stratified" regions in this context can either correspond to regions with weak temperature stratification (small N), or in rapid rotation, or with small Prandtl number—this distinction will be used later.*

FIG. 15: Garaud and Brummell 2009

87. Kristen on the mechanism of MC penetration in stars. I understand that we cannot do a causality analysis for which comes first, the meridional circulation or the horizontal temperature gradient, but how do we understand it? Thermo-viscous: two

competing effects, that's why the prandlt number. I see how viscosity transfers momentum, but I still have a hard time with thermal/radiative diffusivity and its role. How is it different from conduction? How is it relevant for the atmosphere and in stars?

Stars are very nice heat diffusers because of photons which carry heat very efficiently. This is not true for planets. That is how radiative diffusion is different from molecular/thermal diffusion. Radiation carries heat way more efficiently than molecules and the latter is the one happening in planets.

Anyway, coming to torquing of the stellar surface or more specifically the radiative zone. The radiative zone has a sub-adiabatic (stable) temperature gradient which is set by a steady radiative diffusion of heat from the core to the surface. Initially, we even assume the velocity perturbations to be zero in the radiative zone. As the torque starts acting the first and the fastest process that takes place is geostrophic (adiabatic - S&Z) adjustment. This process is purely dynamical. Neither viscosity nor radiative diffusion plays any role in it. The timescale is of the order of the rotation period of the radiative zone. The depth of this layer depends on rotation and stratification only and NOT the Prandlt number!

Once this happens, a horizontal temperature gradient develops due to a vertical zonal velocity gradient (thermal wind). So, hotter poles and cooler equator. And these temperatures are still different from that further in the interior. Radiative diffusion sets into action and propagates the horizontal temperature gradients further down. This temperature gradient creates meridional circulation.

So where is the viscous or dragging effect? It acts on the shear. It tries to stop the penetration of the meridional circulation by reducing vertical gradients of zonal velocity. This is why the radiative diffusion and viscous drag times appear on different sides of our 1D equation.

This longer process takes into account the Prandlt number in addition to rotation and stratification and the zonal velocity penetrates to a depth where there is a balance between drag and radiative diffusion.

We may thus conclude that if the horizontal eddy-diffusivity v_H is large enough, the spread of the tachocline will be limited to a thickness fixed by the first zero of the lowest mode, which is located at $\zeta = 3\pi/4$, that is

$$h = \frac{3\pi}{4\mu_4} d = \frac{3\pi}{2\mu_4} r_0 \left(\frac{\Omega}{N}\right)^{1/2} \left(\frac{\kappa}{v_H}\right)^{1/4}, \quad (5.19)$$

where $\mu_4 = 4.933$ is the smallest non-zero eigenvalue of (5.6). The horizontal turbulence then enforces a stationary state, in which the advection of angular momentum is balanced by the Reynolds stresses acting on the horizontal shear. Such a stationary solution is displayed in fig. 3, where we have arbitrarily set $h = 50,000$ km.

FIG. 16: Spiegel and Zhan on horizontal turbulence.

88. How does the torque from the top boundary transfer zonal momentum to the inner radiative fluid at $t=0$? Is it not through molecular/turbulent viscosity?

Nope! Not at all. It is through a purely mechanical process called geostrophic/adiabatic adjustment, in-

dependent of viscosity and radiative diffusion. See figure 5.

89. Why should there be adiabatic/geostrophic adjustment if geostrophic balance is already satisfied in S & Z equations?

Right! The final 1D equation we have, or even S&Z's does captures what happens after fast adiabatic and thermal adjustments. Our 2D equations, however, shows the thermal adjustment (unsteady term for temperature perturbations present) but definitely assumes geostrophic balance.

90. What does fast thermal relaxation mean? This is possibly related to the adiabatic adjustment of S and Z, right? What kind of physics do we eliminate by setting the unsteady temperature term to zero i.e. by fast thermal relaxation?

Adiabatic adjustment is NOT the same as fast thermal relaxation according to S & Z. They happen on different timescales. The first on the order of the rotation period, the second on the order of $h^2 \rho C_p / \kappa$, where h is the thickness of the tachocline or the layer over which the vertical temperature gradient exists. The thermal adjustment is also small because h is small initially and thereafter a balance of buoyancy and advection is maintained; see figure 4.

We eliminate sound, inertial and gravity waves by setting time derivatives of density, horizontal velocity and vertical velocity to zero. **Yet to figure out what kind of oscillations are cut out because $\partial T / \partial t = 0$. Thermal oscillations?**

91. A simple way to understand Eddington-Sweet circulation.

Rotation makes the star oblate. The radius at the equator is larger than at the poles. Considering the equation for thermal equilibrium for a star, the temperature gradient set by radiative diffusion will result in a lower temperature at the surface at the equator compared to the poles just because it is farther from the core. This will result in baroclinicity and cause meridional circulation.

92. Can we solve a nonlinear ordinary differential equation as an eigenvalue problem?

In general, it is not possible to solve a nonlinear ordinary differential equation (ODE) as an eigenvalue problem, because eigenvalue problems are typically associated with linear operators.

Nonlinear ODEs, on the other hand, involve nonlinear functions of the dependent variable and its derivatives. These equations generally do not have the structure of an eigenvalue problem, and they require different techniques for their solution.

However, there are some special cases where a nonlinear ODE can be transformed into an eigenvalue problem. For example, certain types of nonlinear ODEs can be reduced to a Sturm-Liouville eigenvalue problem, which involves finding eigenvalues and eigenfunctions of a second-order linear differential equation.

93. Can we solve a system of linear partial differential equations as an eigenvalue problem?

Yes, it is possible to solve a system of linear partial differential equations (PDEs) as an eigenvalue problem. In fact, eigenvalue problems are often used to solve linear PDEs.

Eigenvalue problems arise when studying the behavior of linear operators, which are mappings between

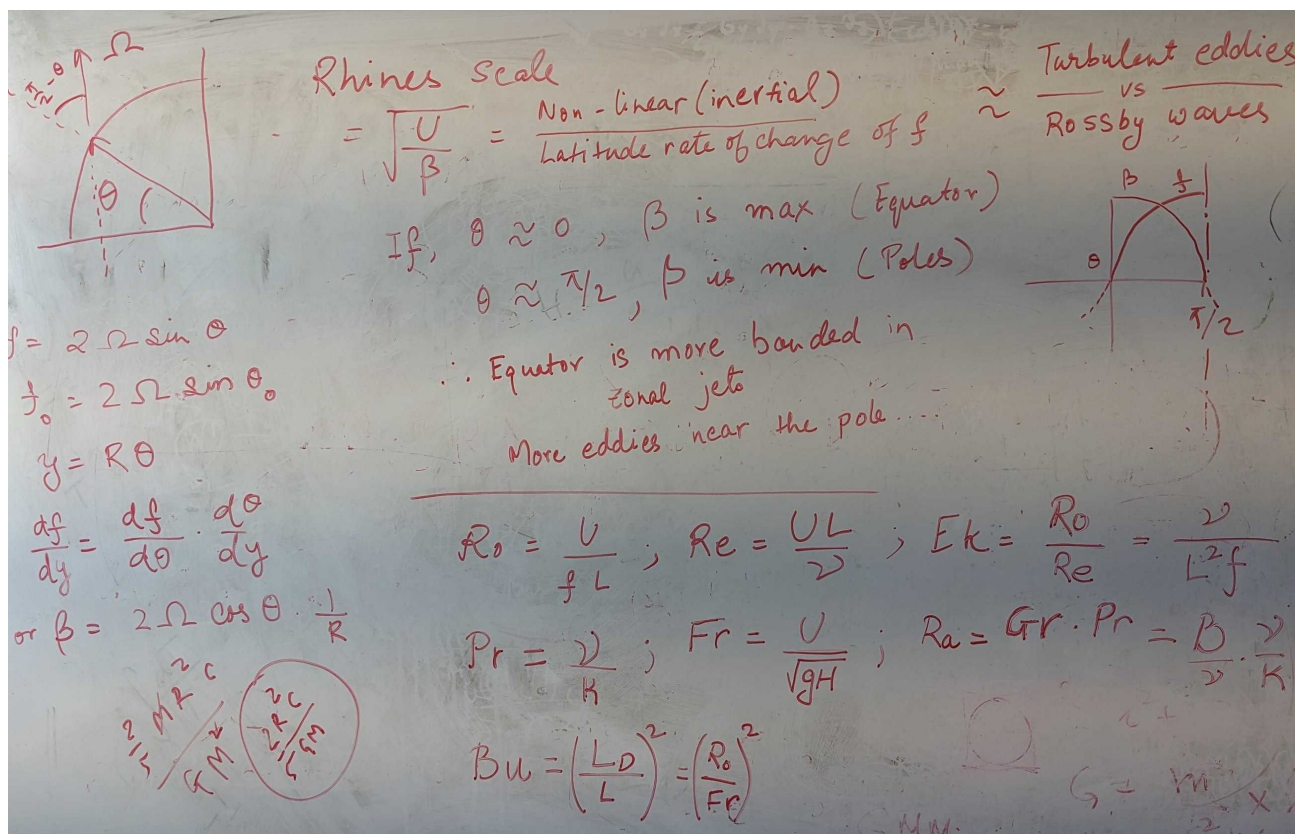


FIG. 17: Nondimensional numbers.

functions. Many PDEs can be represented as linear operators, and the solutions to these PDEs can be found by solving the associated eigenvalue problems.

For example, consider the following system of linear PDEs:

$$Lu(x, y) = \lambda u(x, y)$$

where L is a linear differential operator, $u(x, y)$ is the unknown function, and λ is the eigenvalue. If we can find a set of eigenvalues λ and corresponding eigenvectors $u(x, y)$, we can use them to construct a general solution to the PDE. The general solution will be a linear combination of the eigenvectors, where the coefficients of the linear combination are determined by the initial or boundary conditions.

94. Can we solve non-homogeneous PDEs as an eigenvalue problem?

No. This is because eigenvalue problems involve homogeneous linear differential operators, while non-homogeneous PDEs have non-zero source terms that are not accounted for in the eigenvalue problem.

95. Let us say that we are solving a system of linear homogenous PDEs as an eigenvalue problem. What are is the equivalent of the matrix here? What do the eigenvalues and eigenvectors imply?

The equivalent of the matrix is the linear differential operator that defines the system of PDEs. This operator can be represented in matrix form if we discretize the PDEs using a finite difference or finite element method.

The eigenvalues determine the rate at which the solutions oscillate or decay over time, and the eigenvectors determine the spatial distribution of the oscillations or decays. The eigenvectors are also used to construct a general solution to the PDE system,

which is a linear combination of the eigenvectors with coefficients determined by the initial or boundary conditions.

Furthermore, the eigenvalues and eigenvectors can provide insight into the behavior of the PDE system. For example, the eigenvalues can indicate whether the solutions are stable or unstable, and the eigenvectors can indicate which modes of oscillation or decay are most dominant.

96. With respect to the above, we may also obtain a matrix by looking for wave-like solutions to the system of PDEs. Please elaborate on that.

To obtain matrix A , we can look for wave-like solutions to the system of PDEs, which are solutions of the form $u(x, t) = \exp(i(kx - \omega t))$ that satisfy the PDEs. By substituting this form into PDEs linearized about some equilibrium solution, we can obtain a set of algebraic equations for the coefficients of the wave solution. These equations can then be used to construct the matrix A , which will depend on the wave number k and the frequency ω .

97. Thus, linear differential equations can be reduced to linear algebraic equations by either trying to solve them numerically using finite elements or spectral methods or be plugged in with wave-like solutions to generate an EVP determining the stability of certain basic states.

98. Rhines scale versus Rossby deformation radius. Rhines scale ($\sqrt{U/\beta}$) = nonlinear or inertial (eddy) / beta (Rossby wave) \rightarrow width of zonal jets
 Rossby deformation = NH / f \rightarrow scale of generation of baroclinic eddies

99. Is the beta effect larger near the poles or the equator?

The beta effect is the variation in the Coriolis parameter (f) with latitude in a rotating reference frame. Specifically, the beta effect is given by:

$\beta = (2\omega \cos \phi)/R$, where ϕ is the latitude. The magnitude of the beta effect decreases with increasing latitude, so it is larger near the equator than at the poles.

100. What is the relationship between rotation and bandedness of 2-D zonal flows?

2-D turbulence organises into a tight zonal structure with jet width equal to the Rhines scale. Near the equator where β is large, the jet width is smaller and hence it is more banded (vortices merge along latitudes more than they do along longitudes). As we move towards the poles, β decreases and the structure of the flow is more eddy like. This is subject to the condition that the velocity forcing is the same everywhere.

Note that the polar flow is in greater geostrophic balance than the equator.

101. When is the Cauchy stress tensor asymmetric?

The stress tensor need not be symmetrical for complex fluids, i.e. fluids with a non-trivial microscopic structure. Nematic liquid crystals are an archetypal example of such materials.

102. What is bulk viscosity?

The Bulk Viscosity. The bulk viscosity μ_B of a fluid is also known as the volume viscosity or the second viscosity. It is often not used in classical CFD applications but it plays a role in acoustics as well as high-velocity compressible flows.

103. How is linear relaxation capturing effects of diffusion in both y and z ? Derive how linear relaxation is the first order term in the expansion of the diffusion operator.

104. What is the effect of the meridional periodicity of the forcing velocity on the physics of evolution? Could some other meridional profile result in some kind of instability? In other words, can we apply any function of y to the forcing and expect it to work?

Has to be periodic for dedalus to work, but we could have any periodic function actually. Gyroscopic pumping will work so long the Coriolis force changes sign at least once in the domain. For a fully super-rotating zonal wind forcing, the Coriolis force changes sign at least at the equator, causing a converging flow at the equator. The Coriolis force also changes sign when the direction of the forced wind changes, thus allowing for more meridional cells.

105. There are two time scales relevant to our problem. One, time to steady-state of our meridional circulation or the zonal wind profile. Two, time for a particle to circulate through the meridional loop. Then there is the E-S time, which is thermally driven, which is the basis for our descriptions. What are the relative magnitudes for the Sun?

Eddington-Sweet time is the mixing time at steady-state. However, the mixing or turn-over time is also a function of time or depth of penetration of the meridional circulation. S&Z says that the depth of penetration changes as $(t/t_{ES})^{1/4}$ and covers the entire radiative zone at $t = t_{ES}$.

See figure 11.

106. Where is angular momentum balance in our equations?

One can try to integrate global angular momentum but it will be hard to prove anything with a boundary forcing (I guess in 2D you can integrate the forcing and integrate the angular momentum separately).

107. What is our equation of state (EOS)? What does it help us solve for?

The ideal gas law. It solves for density. Look up Vallis or Holton.

108. Where does the frictional/drag timescale come from for the sun?

It could be a combination of magnetic drag, reynolds stress and other effects.

109. Are we convinced that 1D treatment cannot say anything explicitly for the effect of mean molecular weight gradients?

We have just derived further equations to say something more specific. **add derivation**

110. Why are implicit methods better at handling discontinuities than explicit methods?

They are unconditionally stable. Timestep size need not be limited by the CFL criterion. Less numerical diffusion than explicit methods. Can naturally handle nonlinearities. Better mass, momentum and energy conservation properties.

111. What are explicit methods good for?

Large computational domains. Computationally more efficient, because no matrix inversions or iterative convergence to solution required. Simple to implement. Easier to debug or even interpret someone else's code.

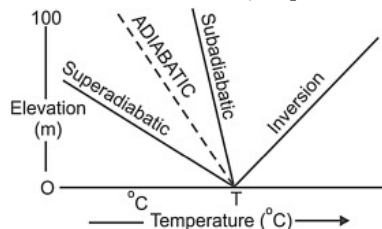
112. What does the advection term in our 1D equation for zonal wind penetration do?

The advection term $k * du/dp$ prevents the circulation from penetrating deep because the sign is negative when brought to the left. Here, 'k' would be some radially outward / vertically upward background velocity advecting the radiatively diffused zonal velocity field outwards. I know it comes from the inhomogeneous diffusion, but what does it imply physically? Is it the vertical component of the meridional circulation? It means it diffuses more upward than it does downwards. The strength of diffusion is location-dependent. All these are valid interpretations.

113. Why is the outer Sun convective even through it is hotter near the core?

How strong a temperature gradient gets, depends on the local efficiency of energy transport inside the star. But for the sun (and other lower mass stars), the surface radiates energy away efficiently, dropping the temperature there compared to the rest of the star, making the outer layers convective.

114. Subadiabatic = stable; Superadiabatic = unstable



115. Is the adiabatic lapse rate different for different atmospheres?

Yes. $\Gamma_d = g/C_p$.

116. What is the importance of Rossby deformation radius? Can it be greater than the radius of the planet/star?

The RDR is the lengthscale of the most unstable baroclinic mode. It can be greater than the radius of a planet or star, in which case the most unstable mode does not fit into the atmosphere and hence is not seen. The rest of the modes are not as unstable and may not lead to generation of eddies via baroclinicity (no turbulence in hot-jupiters because of that). For the Earth, the RDR is 1/4 times its radius.

117. Thermal/radiative forcing conserves angular momentum, so one might see both prograde and retrograde winds developing, while momentum forcing will only add angular momentum implying that there may not be any retrograde winds.
118. What is meant by dynamical core of an atmospheric model?

Atmospheric processes have to be divided into two classes—model dynamics and model physics—not to be confused with dynamics and physics. Model dynamics are processes that can be characterized by the grid-scale mean quantities, while model physics occur in spatial scales smaller than the grid box. Thus, dynamics can be resolved explicitly by the model, while physics cannot.

The part of the model code that describes the model dynamics is called the dynamical core. The dynamical core models the large-scale dynamics, or movement of air, and thermodynamics, or movement of heat, through the atmosphere.

119. $N/2\Omega$ is about $100 \sim 1000$ for stars, planets, and other spherical objects. If it was ~ 1 , then it is a flattened disk.
120. Blackbody (thermal) radiation! Spectroscopy.

A blackbody is an ideal object that absorbs and thus emits all frequencies of radiation at a certain temperature and is in thermodynamic (thermal, mechanical, chemical and radiative) equilibrium. In reality, there can never be a continuous spectrum because quantum mechanics says that energy levels are quantised and only certain states are allowed. Nonetheless, for a system with infinitely many energy states it is possible to absorb and emit at all wavelengths, making it a blackbody. As one increases the temperature of the blackbody, the peak of the spectral energy distribution in wavelength space shifts to the left (towards visible from infrared) and thus we are able to see hot metals glow. Human bodies give off infrared radiation and can be seen in the dark (no reflection) using infrared goggles.

121. What kind of physics goes inside the blackbody?
- Blackbody radiation is primarily linked to the thermal state of an object. Firstly, it needs to be in thermal equilibrium i.e. its temperature cannot change or the energy input must be equal to the energy output in terms of radiation (area under the spectral energy density curve). Secondly, non-radiative pathways of energy transfer such as isomerization, polymerization etc must be ignored.
- Electron clouds in a gas/solid have a certain shape at ground state. When excited with light or energy they change shape and then spontaneously fall back to the ground state or other relatively stable states while emitting light. The electron clouds gives rise

to a dipole moment of individual molecules. Shining light / providing energy changes the dipole moment of the molecules (analogous to the shape change of the cloud). Molecular vibrations, which determine the temperature of the blackbody, is intrinsically coupled to the distribution of electrons around it via momentum/angular momentum conservation and electro-magnetic forces. So thermal emission is nothing different from absorption-emission so to speak.

Important laws are the Stefan-Boltzman law (relates total energy emitted to temperature, fourth power) and Wien's displacement law (relates wavelength of max intensity to temperature, inverse), Planck's radiation law (spectral energy density as a function of wavelength for different temperatures).

Another thing to remember is that bound electrons can emit and absorb only in quantized energy packets, while free electrons can have a continuum of states (also quantized but infinitely many states). Metals and ionized media have loads of free electrons and are likely to have a blackbody spectrum rather than distinct lines at which they absorb or emit. That is why most lines that characterize atmospheres of exoplanets are from non-metallic gaseous molecules.

The Sun is a blackbody that emits light mostly in the visible because it is very hot. Nuclear reactions generate high energy gamma ray photons which carry energy via radiative diffusion heating up the gas (transferring energy to free electrons) as it moves through it causing it to give off radiation in a continuous spectrum with varying intensities peaking at visible, hence white (this radiation is due to the temperature of the gas itself — indirectly reflected/transmitted/re-emitted from another source — hotter stars are blue, blackbody peak shifts to the left in wavelength at higher temperatures). However, certain molecules in cooler regions of the Sun (atmosphere?) are able to absorb some of the frequencies resulting in the solar spectrum we see. This tells us about the composition of a star.

Infrared (low energy, low frequency) - molecular vibrations (related to temperature) Visible/UV (high energy, high frequency) - electronic structure

122. What is meant by "explicit gray radiative transfer"? Does it mean the full radiative transfer scheme as compared to linear relaxation?
- Explicit radiative transfer means solving the differential equations of radiative transfer through the atmosphere explicitly (radiative diffusion or linear relaxation are just approximations to that). Grey means there is only one wavelength (in general, one needs to solve at every wavelength of radiation...very compute intensive).
123. What determines the total thickness of the atmosphere? Just the scale height or multiples of it?
- Evelyn: Depends on what you mean by total thickness. The scale height describes how the pressure drops with altitude, and it depends on temperature, gravity, and the mean molecular weight of the atmosphere. So that will affect how vertically extended the atmosphere looks.
- If two planets have the same scale height, but one has a greater surface pressure, then that one would appear thicker because it would take more scale heights to get to a really transparent layer.

If two planets have different masses but the same surface pressure and atmospheric composition, then the less massive planet will have the taller atmosphere because its scale height is lower. Because H depends on gravity.

124. What is a gauge condition? Some examples? Pressure Poisson equation? How to deal with it?

A gauge condition is an additional constraint in a system of PD equations that may produce non-unique solutions otherwise.

For example, incompressible flows. The continuity equation is the divergence-free condition on velocity. The pressure field is not uniquely determined by the momentum equations (gauge pressure = pressure relative to atmospheric pressure). One needs a gauge condition to find a unique solution that satisfies reality. Without it, the solution might be correct, but not physical.

The Pressure Poisson Equation is the gauge condition for incompressible flows. It can be obtained by taking the divergence of the momentum equations and then substituting the continuity equation.

Incompressible flows do not need this because there is an equation of state and energy conservation, which leads to unique solutions.

Gauge conditions could be related to the fact that pressure acts as a Lagrange multiplier in an incompressible flow, which can take several values.

125. How do you find the equilibrium temperature of a planet? <https://www.astro.princeton.edu/~strauss/FRS113/writeup3/#:~:text=The%20equilibrium%20temperature%20of%20a,not%20on%20its%20own%20size.&text=This%20is%20below%20the%20freezing,oceans%20would%20be%20largely%20frozen>

126. What is meant by the free atmosphere?

The portion of the earth's atmosphere above the planetary boundary layer, in which the effect of surface friction on air motion is negligible, and in which the air behaves like an ideal fluid. Heat budget.

127. How does composition affect atmospheric dynamics?

A higher molecular weight implies fewer molecules at a given pressure, resulting in a greater temperature change rate or, equivalently, a smaller radiative timescale. A higher molar heat capacity suggests that the atmosphere is more difficult to heat and cool, leading to a larger radiative timescale. For example, the radiative timescale in a hydrogen atmosphere is more than 15 times longer than the CO₂ atmosphere, solely due to the effects of μ and c_p .

128. Why is the mass conservation equation not the same as the material derivative of density ρ is zero?

In the most general form, the mass conservation equation includes thermodynamically induced changes to density due to expansion, contraction, or heating of the material element, which makes density an active quantity. However, it is true for mass contained in a particular volume. Thus, the material derivative of mass $D(\rho\Delta V)/Dt$ is actually zero in the most general form.

129. Hayley: Calculating the actual CC curves isn't too hard, so I attached the script I've used to display the curves. Apologies for the messiness of the script, so feel free to ask me questions about what's all in there. It's set up to read in spectra for 0G, 3G, and uniform models, calculate the cross correlation

curves at each provided phase, and plot the curves with a vertical offset. You probably don't need everything in the script, but the functions are at the beginning and I'll also point you to:

Line 96: Calling the CC function and returning the cross correlation curve

Line 116: Finding the maximum of said CC function

Line 120/121: plotting CC function and its maximum with a vertical offset

130. Hayley: On CCFs! You're right for the Doppler off! More specifically, the chosen phase will rotate the 3D temperature structure so that the viewing angle is correct to calculate the spectra from. For chemical composition, we assume solar abundances and LTE. (This can be altered though!) I also make the choice of what opacity sources to include, which for UHJs are usually: CO, H₂O, TiO, VO, K, and Na (but this is also flexible).

131. How are the peaks of CCFs generated?

132. Why does the Sun rotate?

<https://physics.stackexchange.com/questions/547446/why-does-the-sun-rotate>

133. On optical depth τ .

Optical depth is nondimensional length, and the parameters that nondimensionalize, contain optical properties of the media through which radiation propagates. $d\tau = k_\nu \rho ds$. k_ν is the extinction coefficient of the medium (mostly gas). $\tau = 1$ is thus the photosphere for all wavelengths, but they occur at different true depths from the surface of a star or planet. That is why τ has a subscript for frequency, τ_ν .

134. What is meant by a grey radiative transfer scheme?

Optical depth is independent of ν . The radiative transfer equations can be simply integrated over frequencies to get a general form for longwave and shortwave.

$$\frac{dD}{d\tau} = B - D; \frac{dU}{d\tau} = U - B. \quad (2)$$

These are the *two-stream* grey equations for atmospheres. $B = \sigma T^4$ is the Stefan-Boltzmann law! B is the radiative flux due to a blackbody at temperature T . U and D are frequency-integrated upward and downward spectral irradiance/intensity. B and U naturally have the same units and are descriptions for energy flux, per unit area, per unit solid angle.

135. What is the reason for the radius valley?

Two main drivers for the mass loss that sculpts the sub-Neptune population have been proposed: "photoevaporation" (Owen & Wu, 2013) and "core-powered mass loss" (Ginzburg et al., 2018). These two models share a similar physical basis: heating of the planet's upper atmosphere drives a hydrodynamic outflow, akin to a Parker wind (Parker, 1958), resulting in mass loss. However, the energy source that provides the heating of the upper atmosphere and drives the outflow differs. In the photoevaporation model, high-energy, ionizing, extreme ultraviolet (XUV) photons produced in the stellar corona are absorbed by the planet's upper atmosphere. Due to the destruction of molecular coolants by the ionizing photons, the upper atmosphere it is heated to high temperatures (a few thousand to 104 K), driving a hydrodynamic outflow (García

Row	Article	r/R_{\odot}	Tach. thick. (Km)	Assumptions	Remarks
1	[2]	0.314	270,000	Only rad. diff., small tach. thickness, integrate 4.10	Current
2		0.44	180,000	Only rad. diff., integrate 4.9	Current
3		0.08	434,000	Horizontal turb.	Steady state calc
4		0.67	21,000	$Pr = 1$, Hor. turb.	SS calc
5	This article	0.625	5,500	$\tau_R = 9e14s$, $\tau_F = 3e20s$	Current
6		0.414	200,000	$Pr = 1$	Steady state
7		0.0	490,000	$f/N\sqrt{Pr}=0.89$	Steady state
8	[3]	0.55	105,000	Observation, variable $f/N\sqrt{Pr}$	Current
9	[4]	0.68	14,000	Observation	Current

TABLE I: Fractional distance of the lower boundary of the tachocline from the center of the Sun for various models of the tachocline and their corresponding assumptions.

Muñoz, 2007; Lammer et al., 2003; Murray-Clay et al., 2009; Owen & Jackson, 2012; Yelle, 2004). In the core-powered mass-loss model, heating of the upper atmosphere from infrared (IR) radiation from the cooling planetary interior and bolometric irradiation from the star similarly drives a hydrodynamic outflow, albeit a cooler and slower one.

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